Álgebra 2 20/10 4

Dado X um conjunto $f : X \longmapsto X$ f uma bijeção (uma permutação de X)

$Sym(X) = (\{f \mid f \text{ perm de } X\}, \circ)$ ↳ composição

Vimos que é um grupo

Seja $\mathbb{F} \subset \mathbb{K}$ uma <u>extensão de corpos</u> ou seja $\mathbb{F}$ é um corpo do corpo $\mathbb{K}$.

$$\begin{array}{c} \mathbb{K} \\ | \\ \mathbb{F} \end{array}$$

$[\mathbb{K} : \mathbb{F}] = \dim_{\mathbb{F}}(\mathbb{K})$

uma $\varphi : \mathbb{K} \longmapsto \mathbb{K}$ é dita um $\boxed{\mathbb{F}\text{-automorfismo}}$ de $\mathbb{K}$ se

1) $\varphi(x+y) = \varphi(x) + \varphi(y)$
$\varphi(x \cdot y) = \varphi(x)\varphi(y)$
$\forall\, x, y \in \mathbb{K}$ (Homomorfismo)

2) φ é sobrejetora

3) $\forall x \in \mathbb{F}$ $\varphi(x) = x$ ou seja

$\varphi|_{\mathbb{F}} = Id_{\mathbb{F}}$ — $\boxed{\text{F - automorfismo}}$

$$Aut(\mathbb{K}/\mathbb{F}) := \{\varphi \mid \varphi \text{ é F-AUTom. de } \mathbb{K}\}$$

com a operação de composição

__Lema__ $(AUT(\mathbb{K}|_{\mathbb{F}}), \circ)$ é um grupo ∎

__Exemplos__

$\mathbb{F} = \mathbb{R}$

$\mathbb{K} = \mathbb{Q}(\sqrt{2}) = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$

$Aut(\mathbb{K}|_{\mathbb{F}}) = ?$

1°) $\varphi \in \operatorname{Aut}(\mathbb{K}|\mathbb{F})$

$\varphi(1) = \varphi(1\cdot 1) = \varphi(1)\cdot\varphi(1) = [\varphi(1)]^2$

$\varphi(1)[\varphi(1) - 1] = 0 \Rightarrow$

$\varphi(1) = 0$ ou $\boxed{\varphi(1) = 1}$

Se acontecer $\varphi(x) = \varphi(1, x) = \varphi(1)\cdot\varphi(x) =$

$= 0 \ \forall x \in \mathbb{K} \Rightarrow \varphi \equiv 0$, Abs

$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 2$

$\vdots$

$\varphi(n) = n \ \forall n \in \mathbb{N}$

$\varphi(0) = \varphi(0+0) = \cancel{\varphi(0)} + \varphi(0)$

$$\boxed{\varphi(0) = 0}$$

$0 = \varphi(0) = \varphi(n + (-n)) = \varphi(n) + \varphi(-n)$

$\varphi(-n) = -\varphi(n) = -n \ \forall n \in \mathbb{N}$

Daí $\varphi(z) = z \ \forall z \in \mathbb{Z}$

$\forall n, m \in \mathbb{Z}, m \neq 0$

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$$\varphi\left(\frac{n}{m}\right) = \varphi\left(n \cdot \frac{1}{m}\right) = \varphi(n) \cdot \varphi\left(\frac{1}{m}\right) =$$

$$= n \cdot \varphi\left(\frac{1}{m}\right) = n \cdot \frac{1}{m} = \frac{n}{m}$$

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$$1 = \varphi(1) = \varphi\left(m \cdot \frac{1}{m}\right) = \varphi(m) = \varphi\left(\frac{1}{m}\right)$$

$$= m\varphi\left(\frac{1}{m}\right) \Rightarrow \varphi\left(\frac{1}{m}\right) = \frac{1}{m}$$

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Logo

$$\boxed{\varphi(q) = q \ \forall q \in \mathbb{Q}}$$

Tome $\alpha \in \mathbb{K} \Rightarrow \boxed{\exists\ x, y \in \mathbb{Q}}$

t.q.

$$\alpha = x + y\sqrt{2}$$

Daí $\varphi(\alpha) = \varphi(x) + \varphi(y) \cdot \varphi(\sqrt{2})$

$$= x + y\,\varphi(\sqrt{2})$$

$$[\varphi(\sqrt{2})]^2 = \varphi(\sqrt{2}) \cdot \varphi(\sqrt{2}) = \varphi(\sqrt{2} \cdot \sqrt{2})$$

$$= \varphi(2) = 2$$

Logo $[\varphi(\sqrt{2})^2] = 2 \Rightarrow$

$\varphi(\sqrt{2}) = \sqrt{2}$ ou $\boxed{\varphi(\sqrt{2}) = -\sqrt{2}}$

$$\boxed{\varphi(\sqrt{2}) = \sqrt{2}}$$

$$\varphi_1(\alpha) = x + y\sqrt{2} = \alpha \qquad \varphi_1 = Id$$

$$\boxed{\varphi(\sqrt{2}) = -\sqrt{2}}$$

$$\varphi_2(\alpha) = x - y\sqrt{2}$$

$$\varphi_2^2(\alpha) = \varphi_2(\varphi_2(\alpha)) = \varphi_2(x - y\sqrt{2}) =$$

$$= \varphi_2(x) - \varphi_2(y) \cdot \varphi_2(\sqrt{2}) = x - y \cdot (-\sqrt{2})$$

$$= x + y\sqrt{2} = \alpha \quad \forall \alpha \in \mathbb{K}$$

Logo $\varphi_2^2 = Id = \varphi_1$

$Aut(\mathbb{K}|_{\mathbb{F}}) = \{\varphi_1, \varphi_2 \mid \varphi_2^2 = \varphi_1\} \cong C_2$

Exercício

Calcule $Aut(\mathbb{C}|\mathbb{R}) = \{\varphi_1, \varphi_2\}$

$\varphi_2(z) = \overline{z}$ & $\varphi_1 = Id$

$\varphi_2^2 = \varphi_1$

$\varphi_2(z) = z$
$\forall z \in \mathbb{R}$

$\varphi \in Aut(\mathbb{K}|_{\mathbb{F}})$ & $a_0, \dots, a_n \in \mathbb{F}$
& $f(x) = a_0 + a_1 x + \dots + a_n x^n$

$\boxed{\text{Se } \alpha_0 \in \mathbb{K} \text{ é uma raiz da } f}$

então $\varphi(\alpha)$ também é raiz da f.

Temos $f(\alpha_0) = 0$ $a_0 + \dots + a_n \alpha_0^n = 0$

Aplique φ à eq. lembrando que

$\varphi(a_i) = a_i \ \forall i$ & $\varphi(z^l) = [\varphi(z)]^l$

$\forall z \in \mathbb{K}$

$0 = a_0 + a_1 \varphi(\alpha_0) + \dots a_n [\varphi(\alpha_0)]^n =$

$= f(\varphi(\alpha_0))$ Logo $\varphi(\alpha_0)$ é raiz

da f

$\mathbb{K} = \mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\underbrace{(\sqrt[3]{2})^2}_{\sqrt[3]{4}} \mid$

$a, b, c \in \mathbb{Q}\}$

$\mathbb{F} = \mathbb{Q}$

$\mathrm{Aut}(\mathbb{K}|_{\mathbb{F}}) = ?$

$\varphi \in \mathrm{Aut}(\mathbb{K}|_{\mathbb{F}})$

$\alpha = a + b\sqrt[3]{2} + c\sqrt[3]{4}$, $a, b, c \in \mathbb{Q}$

$\varphi(\alpha) = a + b\varphi(\sqrt[3]{2}) + c[\varphi(\sqrt[3]{2})]^2$

Logo se souber "o valor" de

$\varphi(\sqrt[3]{2})$ conheço a φ

$$[\varphi(\sqrt[3]{2})]^3 = \varphi((\sqrt[3]{2})^3) = \varphi(2) = 2$$

$$[\varphi(\sqrt[3]{2})]^3 = 2$$

ou seja $\varphi(\sqrt[3]{2})$ é raiz de $f(x) =$

$= x^3 - 2 \in Q[x]$

$\boxed{\sqrt[3]{2}}$ raízes de f

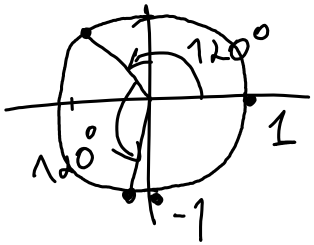
$$x^3 = 2$$

$$\frac{x^3}{2} = 1 \qquad \left(\frac{x}{\sqrt[3]{2}}\right)^3 = 1 \Rightarrow \frac{x}{\sqrt[3]{2}} \text{ é}$$

uma

raiz terceira da unidade que são

$$1,\ \underbrace{e^{2\pi i/3}}_{= \xi_3},\ \underbrace{e^{4\pi i/3} = (e^{2\pi i/3})^2}_{= \xi_3^2}$$



$\xi \notin \mathbb{R}$

Daí $\dfrac{x}{\sqrt[3]{2}} \in \{1, \xi_3, \xi_3^2\}$

Logo $\mathbb{R}_{x^3-2} = \{\sqrt[3]{2}, \xi_3\sqrt[3]{2}, \xi_3\sqrt[3]{2}\}$

é o conjunto de raízes de $f(x) = x^3 - 2$

Logo $\varphi(\sqrt[3]{2})$

- $\boxed{\sqrt[3]{2}}$
- $\xi_3\sqrt[3]{2} \notin \mathbb{R}$
- $\xi_3^2\sqrt[3]{2} \notin \mathbb{R}$

$\varphi(\sqrt[3]{2}) \in \mathbb{R}$

$\mathbb{K} \subseteq \mathbb{R} \Rightarrow \varphi(\sqrt[3]{2}) \in \mathbb{K} \subseteq \mathbb{R}$

Logo $\varphi(\sqrt[3]{2}) = \sqrt[3]{2}$

Daí $\varphi(\alpha) = a + b\sqrt[3]{2} + c\left(\sqrt[3]{2}\right)^2 = \alpha$

$\varphi = Id.$

$$Aut(\mathbb{K}_{|\mathbb{F}}) = \{Id\}$$

Lema

Seja $\mathbb{K}_{|\mathbb{F}}$ extensão de corpos

$$n = [\mathbb{K} : \mathbb{F}] \in \mathbb{N}$$

então $|Aut(\mathbb{F}_{|\mathbb{K}})| \leq n$

$F = \mathbb{Q}$

$$\mathbb{K} = \mathbb{Q}(x) = \left\{ \frac{f}{g} \;\middle|\; f, g \in \mathbb{Q}[x] \right\}$$

$g \neq 0$

uma função racional

$Aut(\mathbb{K}_{|\mathbb{F}}) \ni \varphi$

$$\mathbb{K} \ni \alpha = \frac{a_0 + \dots + a_n x_n}{b_0 + b_1 x + \dots + a_m x^m} \begin{matrix} \leftarrow f \\ \leftarrow 0 \neq g \end{matrix}$$

$$\varphi(x) = \frac{a_0 + a_1 \varphi(x) + \dots + a_n [\varphi(x)]^n}{b_0 + \dots + b_m [\varphi(x)]^m}$$

Logo basta eu saber "o valor" de

$$\boxed{\varphi(x)}$$

$\varphi(x) = x \iff \varphi = Id$

$\varphi(x) = -x \quad \varphi = \varphi_2$

$\varphi_2^2 = Id$ mas $\varphi_2 \neq -Id$

$\hookrightarrow \not\equiv Aut(K|F)$

$\varphi(x) = \frac{1}{x}$

$\forall\, a, b, c, d \in Q$ com $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$

$$\boxed{\varphi(x) = \frac{ax+b}{cx+d}} \to \text{Transformação de } \cancel{\text{Mobius}} \text{ Möbius}$$

serve

(Funções Analíticas)

$n = [\mathbb{K} : \mathbb{F}] \in \mathbb{N}$

$[\mathbb{Q}(x) : \mathbb{Q}] = ?\ \infty.$

1) Número alg & Número transcendentes

não cai