ESTOC 2/3

Exponencial (α)

$$f_X(x) = \alpha e^{-\alpha x}, \quad \alpha > 0, \; x \geq 0$$

$$E(X) = \frac{1}{\alpha} \qquad V(X) = 1/\alpha^2$$

Propriedade Falta de Memória

$$P(X > t + r \mid X > t) = P(X > r)$$

$$P(X > u) = \int_u^{\infty} \alpha e^{-\alpha x}\, dx = -e^{-\alpha x} \Big|_u^{\infty} = e^{-\alpha u}$$

$$P(X > t + r \mid X > t) = \frac{P(X > t + r)}{P(X > t)} =$$

$$= \frac{e^{-\alpha(t+r)}}{e^{-\alpha t}} = e^{-\alpha r} = P(X > r).$$

Distribuição acumulada

X é uma v.a.

$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$

$$F'_X(x) = f_X(x)$$

Propriedades:

(i) $F(.)$ é não decrescente

(ii) $\lim_{x \to +\infty} F(x) = 1$ $\lim_{x \to -\infty} F(x) = 0$

(iii) $P(a < X < b) = F(b) - F(a)$

xercício

Seja U uma v.a. uniforme contínua em $[0, 1]$. Qual a distribuição ou densidade de $X = -\ln U$?

$$F_X(x) = P(X \leq x)$$
$$= P(-\ln U \leq x)$$
$$= P(-\ln U \geq -x)$$
$$= P(U \geq e^{-x}) = 1 - e^{-x}$$

$$f_X(x) = e^{-x}, \quad x \geq 0$$

exponencial (1)

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Distribuição Conjunta

Y \ X	0	2	4
-1	1/8	0	1/8
0	0	2/8	0
1	2/8	0	2/8

$\mathbb{P}(\omega : X(\omega) = 0, Y(\omega) = -1) = 1/8$

$-(x,y)$ densidade conjunta

$x,y)$

$$= (x,y) > 0 \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

(x,y)

$$P(X=0) = \sum P(X=0, Y=y)$$

$$=_x (x) = \int_0^0 f(x,y)\,dy$$

$$f_Y(y) = \int f(x,y)\,dx$$

$$F_{X,Y}(a,b) = P(X \le a, Y \le b) \quad \text{Rab}$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} F(x,y)\,dy\,dx$$

$$f(x,y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} F(x,y)$$

Condicional e Independência

$$f_{X/Y=y}(x) = \frac{f(x,y)}{f_Y(y)} \; ; \; f_Y(y) > 0$$

X e Y são independentes se e só se

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

<u>Exemplo 1</u> $f(x,y) = \begin{cases} 6x^2y, & 0<x<1; 0<y<1 \\ 0 & \text{c.c.} \end{cases}$

$$\int_0^1 \int_0^1 6x^2 y \, dy \, dx = \int_0^1 6x^2 \frac{y^2}{2} \Big|_0^1 dx =$$

$$= \int_0^1 3x^2 \, dx = x^3 \Big|_0^1 = 1$$

$$f_X(x) = \int_0^1 f(x,y)\,dy$$

$$= 3x^2 \quad 0 < x < 1$$

$$f_Y(x) = \int_0^1 f(x,y)\,dx = y\,[a x^3]\Big|_0^1 =$$

$$= 2y, \; 0 < y < 1$$

"qqer função que a área ser 1, de 0 a 1, é densidade

$$f_{X/Y=y}(x) = \frac{6x^2y}{2y} = 3x^2$$

<u>**Exemplo 2:**</u>

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0 & c.c. \end{cases}$$

$$f_X(x) = \int f(x,y)\,dy = \int_x^1 2\,dy = 2y\Big|_x^1 =$$

$2 - 2x$, ~~0<x~~ $0 < x < 1$

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1$$

Esperança Matemática

$X \longrightarrow f_X(x)$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$Y = g(X) \quad E(Y) = ?$

$$g(x)\, f(x)\, dx$$

$E(X+Y) = ?$ $\quad \int\int (x+y) \cdot f(x,y)\, dx\,dy$

Distribuição Gama (α, β)

$$f_X(x) = \frac{\alpha^{\beta} e^{-\alpha x} x^{\beta-1}}{\Gamma(\beta)}, \quad x \geq 0$$

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} \, dx$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(a) = (a-1)\,\Gamma(\ , \ .$$

$\beta = 1$ ex (α)

$$E(X) = \int_0^{\infty} x \, \frac{\alpha^{\beta} e^{-\alpha x} x^{\beta-1}}{\Gamma(\beta)} \, dx$$

$$= \int_0^{\infty} \frac{\alpha^{\beta} e^{-\alpha x} x^{(\beta+1)-1}}{\Gamma(\beta)} \, dx$$

$$\Gamma(\beta+1) \int_0^\infty \frac{\alpha^{\beta+1} e^{-\alpha x} x^{(\beta+1)-1}}{\Gamma(\beta+1)} dx =$$

$$= \frac{\overset{\beta\Gamma\cancel{(\beta)}}{\Gamma(\beta+1)}}{\cancel{\Gamma(\beta)}\,\alpha} = \frac{\beta}{\alpha}$$

$$V(X) = \frac{\beta}{\alpha^2}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\alpha^\beta e^{-\alpha x} x^{\beta-1}}{\Gamma(\beta} dx$$

$$= \int_0^\infty \frac{\alpha^\beta e^{-\alpha x} x^{(\beta+2)-1}}{\Gamma(\beta)} dx$$

$$= \frac{\Gamma(\beta+2)}{\alpha^2\,\Gamma(\beta)} \underbrace{\int_0^\infty \frac{\alpha^{\beta+2}\, e^{-\alpha x}\, x^{(\beta+2)-1}}{\Gamma(\beta+2)}\, dx}_{1}$$

$$= \frac{(\beta+1)\,\beta\,\Gamma(\beta)}{\alpha^2\,\Gamma(\beta)} = \frac{\beta(\beta+1)}{\alpha^2}$$

$$V(X) = E(X^2) - E^2(X)$$

$$= \frac{\beta^2}{\alpha^2} - \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2} = \frac{\beta}{\alpha^2}$$

Teorema: Se X e Y são v.a's independentes, exponenciais (α), então

$$X + Y \sim \text{Gama}(\alpha, 2)$$

orolário: X_i iid $\exp(\alpha)$ $\sum_{i=1}^{n} X_i \sim \Gamma(\alpha, n)$