Comp. musical 25/03/09

periódica

$f: \mathbb{R} \longrightarrow \mathbb{R}$

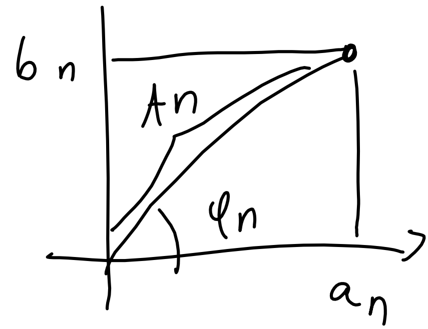
$$f(t) = \sum_{n=0}^{\infty} \underbrace{a_n \operatorname{sen}(\overset{n\omega}{\omega_n} t) + b_n \cos(\omega_n t)}_{A_n \operatorname{sen}(\omega_n t + \varphi_n)}$$

$A_n = \sqrt{a_n^2 + b_n^2}$

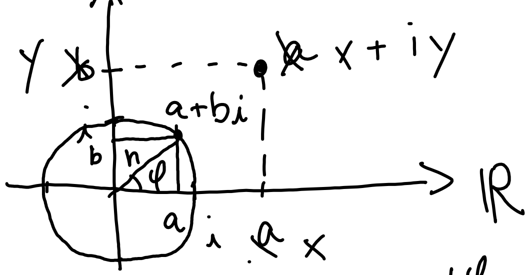
$\varphi_n = \operatorname{arctg} \dfrac{b_n}{a_n}$

$a_n = A_n \cos \varphi_n$

$b_n = A_n \operatorname{sen} \varphi_n$



Plano Complexo



$h = \sqrt{a^2 + b^2} = 1$

$a = \cos \varphi$

$b = \operatorname{sen} \varphi$

$a + ib = \cos \varphi + i \operatorname{sen} \varphi = e^{i\varphi}$ = relação de Euler

Prova da relação de Euler

- Expansão de Taylor

Dada $f: \mathbb{R} \to \mathbb{R}$ encontre $a_0, a_1, a_2, \dots$

tais que $f(x) = \sum_{i=0}^{\infty} a_i x^i$ (*)

subst. $x = 0$: $f(0) = a_0 0^0 = a_0$

Derivando a expr. (*)

$$f'(x) = \sum_{i=0}^{\infty} i a_i x^{i-1}$$

subst. $x = 0 \Rightarrow \boxed{f'(0) = a_1}$

$$f''(x) = \sum_{i=0}^{\infty} i(i-1) a_i \cdot x^{i-2}$$

$x = 0$

$$f''(0) = 2 \cdot 1 \cdot a_2$$

$$a_2 = \frac{f''(0)}{2}$$

A derivada de ordem m de f é: $\qquad \llcorner 0$

$$f^{(m)}(x) = \sum_{i=0}^{\infty} i(i-1)(i-2)\dots(i-m+1)a^{i-m}$$

subst $x=0$, temos

$$f^{(m)}(0) = \overbrace{m(m-1)\dots(1)}^{m!}\,a_m$$

$$\longrightarrow a_m = \frac{f^{(m)}(0)}{m!}$$

derivados de sen(i), cos(i) e e^x

m	$\text{sen}^{(m)} i$	$\cos^{(m)} i$	$[e^x]^{(m)}$
1	$\cos(i)$	$-\text{sen}(i)$	e^x
2	$-\text{sen}(i)$	$-\cos(i)$	e^x
3	$-\cos(i)$	$\text{sen}(i)$	e^x
4	$\text{sen}(i)$	$\cos(i)$	e^x
5	$\cos(i)$	$-\text{sen}(i)$	e^x
	$\vdots$	$\vdots$	

11

$$\operatorname{sen}(x) = \sum_{i=0}^{\infty} a_i x^i = 0x^0 + 1x^1 + 0x^2 +$$

$$\frac{(-1)}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 + \frac{(-1)}{7!}x^7$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \dots$$

$$\cos(x) = \sum_{i=0}^{\infty} b_i x^i = 1 \cdot x^0 - 0x^1 - \frac{1}{2}x^2 + 0x^3$$

$$+ \frac{1}{4!}x^4 + 0x^5 + \frac{(-1)}{6!}x^6$$

$$= 1 - \frac{1}{2}x^{\cancel{3}\,2} + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

$$e^x = \sum_{i=0}^{\infty} a_i x^i = 1x^0 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

<u>Por Taylor:</u>

Por Taylor

L.2

$$e^{i\varphi} = \sum_{m=0}^{\infty} \frac{1}{m!}(i\varphi)^m = 1 + i\varphi + \frac{1}{2} i^2 \varphi^2$$

$$+ \frac{1}{3!} i^3 \varphi^3 + \frac{1}{4!} i^4 \varphi^4 + \dots$$

$$1 + i\varphi - \frac{1}{2}\varphi^2 - \frac{1}{3!} i\varphi^3 + \frac{1}{4!}\varphi^4 +$$

$$- \frac{1}{5!} i \varphi^5 + - \frac{1}{6!}\varphi^6 - \frac{1}{7!} i\varphi^7$$

$$- \left[1 - \frac{1}{2}\varphi^2 + \frac{4}{4!}\varphi^4 - \frac{1}{6!}\varphi^6 + \dots \right]$$

$$+ i \left[\varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 - \frac{1}{7!}\varphi^7 + \dots \right]$$

$$A = \sqrt{x^2 + y^2}$$

$$x + iy = A\cos\varphi + iA\,\text{sen}\,\varphi \;] = Ae^{i\varphi}$$

$$x = A\cos\varphi$$

$$y = A\,\text{sen}\,\varphi$$

$x = a + bi = Ae^{i\varphi}$

$y = c + di = Be^{i\varphi}$

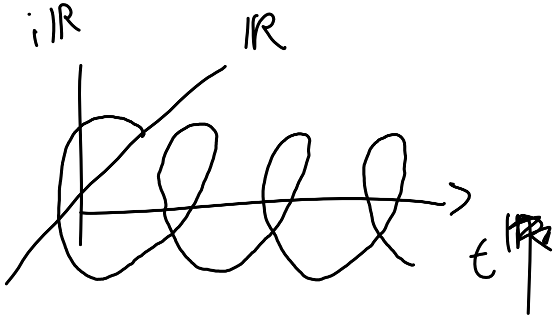
$x + y = (a+c) + i(b+d)$

$xy = (a+bi)(c+di) = ac + bd\,i^2 + i(bc+ad)$

$= ac - bd) + i(bc+ad)$

$xy = (Ae^{i\varphi})(Be^{i\varphi}) = ABe^{i(\varphi+\varphi)}$

$f(t) = e^{i\omega_n t}$



$$e^{i\varphi} = \cos(\varphi) + i\,\mathrm{sen}(\varphi)$$

$$(e^{i\varphi})^* = e^{-i\varphi} = e^{i(-\varphi)}$$

$$= \cos(-\varphi) + i\,\mathrm{sen}(-\varphi)$$

$$= \cos(\varphi) - i\,\mathrm{sen}(\varphi)$$

$$e^{i\varphi} + e^{-i\varphi} = 2\cos(\varphi)$$

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \quad *$$

$$e^{i\varphi} - e^{-i\varphi} = 2i\,\mathrm{sen}(\varphi)$$

$$\mathrm{sen}(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \quad *$$

$$\mathrm{sen}(a)\cos(b) = \left[\frac{e^{ia} - e^{-ia}}{2i}\right]\left[\frac{e^{ib} + e^{-ib}}{2}\right]$$

$$= \frac{e^{i(a+b)} + e^{i(a-b)} - e^{i(b-a)} - e^{-i(a+b)}}{4i}$$

$$= \frac{e^{i(a+b)} - e^{-i(a+b)}}{4i} + \frac{e^{i(a-b) - i(a-b)}}{4i} \quad 15$$

$$= \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$$

$$\cos(a)\cos(b) = \left[\frac{e^{ia} + e^{-ia}}{2}\right]\left[\frac{e^{ib} + e^{-ib}}{2}\right]$$

$$= \frac{e^{i(a+b)} + e^{i(a+b)} - e^{i(b-a)} + e^{-i(a+b)}}{4}$$

$$= \frac{e^{i(a+b)} + e^{-i(a+b)} + e^{i(a-b)} + e^{-i(a-b)}}{4}$$

$$= \frac{\cos(a+b)}{2} + \frac{\cos(a-b)}{2}$$

Dada $f: \mathbb{R} \to \mathbb{R}$ periódica c/ período T (seg) / freq. angular $w = \frac{2\pi}{T}$ rad/seg, encontrar $F_m \in \mathbb{C}$ tais que $f(t) = \sum_{m=-\infty}^{\infty} F_m e^{iw_m t}$

$$f(t) = \sum_{m=-\infty}^{\infty} F_m e^{iw_m t}$$

Ortogonalidade das funções $e^{iw_m t}$:

$$\int_0^T e^{iw_n t} e^{-iw_m t} dt = \begin{cases} 0 & \text{se } n \neq m \\ T & \text{se } n = m \end{cases}$$

<u>Prova</u>:

$$\int_0^T e^{iw_n t} \cdot e^{-iw_m t} dt =$$

$$= \int_0^T e^{i(w_n - w_m)t} dt$$

$$\underset{\substack{se\\ n\neq m}}{=}\left[\frac{e^{i(w_n-w_m)t}}{i(w_n-w_m)}\right]_0^T = \frac{e^{i(n-w)wT}}{i(n-m)w} -$$

$$-\frac{e^{r(n-m)w0}}{i(n-m)w} = 0$$

$$\boxed{17}$$

se $n=m$

$$\int_0^T e^{iw_nt}\,e^{-iw_mt}\,dt$$

$$= \int_0^T e^0\,dt = \int_0^T 1\,dt = T$$

álculo dos F_m's :

A partir da expressão $f(t) = \sum_{m=-\infty}^{+\infty} F_m e^{iw_mt}$

podemos multiplicar por e^{-iw_nt} e integrar :

$$\int_0^T f(t)e^{-w_nt} = \int_0^T \left[\sum_{m=-\infty}^{\infty} F_m e^{+iw\,mt}\right]e^{-iw_nt}\,dt$$

$$= \sum_{m=-\infty}^{\infty} F_m \int_0^T e^{iwmt} e^{-wnt} dt$$

$$= F_n T$$

Eq. de Análise

$$F_n = \frac{1}{T} \int_0^T f(t) e^{-wnt} dt$$

Eq. de Síntese

$$f(t) = \sum_{m=-\infty}^{\infty} F_m e^{iwmt}$$

Propriedade: $F_n = (F_{-n})^*$

Propriedades do conjugado complexo:

$x = a + bi = A e^{i\varphi}$

$y = c + di = B e^{i\psi}$

$$(x+y)^* = (a+c+i(b+d))^*$$
$$= a+c-i(b+d)$$
$$= x^* + y^*$$
$$(xy)^* = (AB\,e^{i(\varphi+\omega)})^*$$
$$= AB\,e^{i(-\varphi-\psi)}$$
$$= A\,e^{-i\varphi}\,B\,e^{-i\psi}$$
$$= x^* y^*$$

<u>Prova</u>: $(F_{-n})^* = \left(\frac{1}{T}\int_0^T f(t)\,e^{-i\omega n t}\,dt\right)^*$

$$= \frac{1}{T^*}\left(\int_0^T f(t)\,e^{j\omega t}_{-n}\,dt\right)^*$$
$$= \frac{1}{T}\int_0^T \left(f(t)\,e^{-i\omega(-n)t}\right)^* dt$$
$$= \frac{1}{T}\int_0^T f(t)^* \left(e^{-i\omega(-n)t}\right)^* dt$$
$$= \frac{1}{T}\int_0^T f(t)\,e^{i\omega(-n)t}\,dt$$

$$-\frac{1}{T}\int_0^T f(t)e^{-i w_n t}\,dt \quad F_n$$

O n-ésimo harmônico de $f(t)$ é dado por (considere $F_n = A_n e^{i\varphi_n}$)

$$h_n(t) = F_n e^{i w_n t} + F_{-n} e^{i w_{(-n)} t}$$

$$= F_n e^{i w_n t} + F_n^* e^{-w_n t}$$

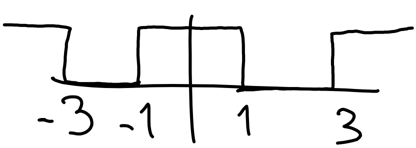
$$= A_n e^{i\varphi_n} e^{i w_n t} + A_n e^{-i\varphi_n} e^{-i w_n t}$$

$$= A_n \left[e^{i(\varphi_n + w_n t)} + e^{-i(\varphi_n + w_n t)} \right]$$

$$= 2A_n \cos(w_n t + \varphi_n)$$

$$\forall n > 0$$

<u>exemplo</u>: Considere o sinal



α1

$= 4 \quad \omega = \frac{2\pi}{T} = \frac{\pi}{2}$

$$F_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{4}\,2 = 1/2$$

$$F_n = \frac{1}{T}\int_{-2}^{2} f(t)\, e^{-i\omega_n t}\, dt$$

$$= \frac{1}{4}\int_{-1}^{2} e^{-i\omega_n t}\, dt = \frac{1}{4}\left[\frac{e^{-i\omega_n t}}{-i\omega_n}\right]_{-1}^{+1}$$

$$= \frac{1}{4}\left[\frac{e^{-i\omega_n}}{-i\omega_n} - \frac{e^{+i\omega_n}}{-i\omega_n}\right]$$

$$= \frac{1}{4}\,\frac{1}{(-i\omega_n)}\cdot\left[e^{-i\omega_n} - e^{i\omega_n}\right]$$

$$= \frac{1}{\cancel{4}\,2}\cdot\frac{1}{(-i\omega_n)}\, -\cancel{2}\cancel{i}\sin(\omega n)$$

$$= \frac{1}{2}\,\frac{1}{\omega_n}\,\text{sen}(\omega_n) = +\frac{1}{2}\,\frac{\text{sen}(n\pi/2)}{n\pi/2}$$ 22

O n-ésimo harmônico é

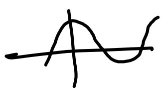
$$h_n(t) = 2A_n \cos(\omega_n t + \varphi_n)$$

$$A_n = |F_n| = \left| \frac{1}{2}\,\frac{\text{sen}(n\pi/2)}{n\pi/2} \right| = \begin{cases} 0, & n \text{ par} \\ \dfrac{1}{n\pi}, & n \text{ ímpar} \end{cases}$$

$$\varphi_n = \text{fase}(F_n) = \begin{cases} 0, & \text{se } n \text{ é par} \\ 0, & \text{se } n = 1, 5, 9, \text{etc} \\ \pi, & \text{se } n = 3, 7, 11, \ldots \end{cases}$$

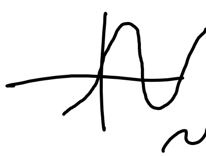
~ // ~

<u>Def</u>: $f : \mathbb{R} \to \mathbb{R}$ é par


cos

se $f(x) = f(-x)\ \forall x$

f é ímpar se $f(x) = -f(-x)$


sen

propriedades de funções pares e ímpares (23)

1) $\sum$ f. pares é par
2) $\sum$ f. ímp. é ímp.
3) f. par × f. par = f. par
4) f. ímpar × f ímpar = f. ~~ímpar~~ par
5) f. par × f ímpar = f. ímpar
6) $\int_{-T}^{T}$ f. par = 2 $\int_{0}^{T}$ f. par
7) $\int_{-T}^{T}$ f. ímpar = 0

Se $\{F_n\}$ é a série de Fourier de uma função $f(\cdot)$, com representações cartesiana e polar dadas por

$$\begin{cases} F_n = a_n + i b_n \\ F_n = A_n e^{i\theta_n} \end{cases}$$

L.4

então

$\{a_n\}$ e $\{A_n\}$

são funções pares do índice n

$\{b_n\}$ e $\{\varphi_n\}$

são funções ímpares de n

Prova: $F_{-n} = F_n^*$

$$a_n + i b_{-n} = a_n - i b_n$$

$$\Rightarrow \quad a_{-n} = a_n$$

$$b_{-n} = -b_n \quad \forall n$$

$$A_{-n} e^{i\varphi_{-n}} = A_n e^{-i\varphi_n}$$

$$\Rightarrow \quad A_{-n} = A_n$$

$$\varphi_{-n} = \varphi_n$$

Toda função f pode ser escrita como

$f_{par} + f_{ímpar}$ onde

$$f_{par}(x) = \frac{f(x) + f(-x)}{2}$$

$$f_{ímpar}(x) = \frac{f(x) - f(-x)}{2}$$

Teorema: Seja $f = f_{par} + f_{ímpar}$, e $\{F_n = a_n + ib_n\}$ a série de Fourier de f. Então $\{a_n\}$ é a série de Fourier de f_{par} e $\{b_n\}$ é a série de Fourier de $f_{ímpar}$

Corolário: f é par $\iff$ $\{F_n\}$ é puramente real (e par).

f é ímpar $\iff$ $\{F_n\}$ é puramente imaginária (e ímpar)

Prova (do teorema).

Se $f(t) = \sum_{m=-\infty}^{+\infty} f_m e^{w_n t}$, considere

$$f_{par}(t) = \frac{f(t) + f(-t)}{2}$$

Então

$$f_{par}(t) = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} F_m e^{i w_m t} + \sum_{m=-\infty}^{\infty} F_m e^{i w_n (-t)}\right]$$

$$= \frac{1}{2} \sum_{m=-\infty}^{\infty} F_m \left[\underbrace{e^{i w_n t} + e^{-i w_m t}}_{2\cos(w_n t)}\right]$$

$$= \sum_{m=-\infty}^{\infty} a_m \cos(w_n t) + \cancelto{0}{i b_m \cos(w_m t)}$$

$$= \sum_{m=-\infty}^{\infty} a_m \cos(w_m t)$$

$$= \sum_{m=-\infty}^{\infty} a_m \cos(w_m t) + i\, a_m \operatorname{sen}(w_m t)$$

$$\sum_{n=-\infty}^{\infty} a_n(\cos(\omega_n t) + i\,\mathrm{sen}(\omega_n t))$$ (2.7)

$$= \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n t}$$

Logo $\{a_n\}$ é a série de Fourier associada a f par